MATH 140A Review: proof by contradiction and the contrapositive

Facts to Know:

To prove a statement by contradiction, do the following:

1. Assume that the statement is false
2. Find the negation of the statement.
3. Use the negation of the statement to find a contradiction.

Example: Show that $\sqrt{2}+\sqrt{3}$ is irrational.

Proof. By way of contradiction, we assume

$$\sqrt{2}+\sqrt{3} \text{ is rational.}$$

Thus,

$$\sqrt{2}+\sqrt{3} = \frac{p}{q},$$

where $p \in \mathbb{Z}$ and $q \in \mathbb{N}$. Note that

$$-1 = 2-3 = (\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3}) = \frac{p}{q}(\sqrt{2}-\sqrt{3}).$$

Thus,

$$(\sqrt{2}-\sqrt{3}) = -\frac{q}{p}.$$

Hence,

$$\frac{p}{q} - \frac{q}{p} = (\sqrt{2}+\sqrt{3}) + (\sqrt{2}-\sqrt{3}) = 2\sqrt{2}.$$

Since

$$\sqrt{2} = \frac{1}{2}\left(\frac{p}{q} - \frac{q}{p}\right),$$

then $\sqrt{2}$ is rational. This is a contradiction since $\sqrt{2}$ is irrational. Thus, $\sqrt{2}+\sqrt{3}$ is irrational.

□

Facts to Know:

$\underbrace{\text{If } P, \text{ then } Q.}_{\text{conditional statement}}$ iff If not Q, then not P.

Examples:

1. Find the contrapositive of: "If you are not happy, then it's time to change something."

   If it's not time to change something, then you are happy.

2. Assume that x is an integer. If $x^{2020} - 3x^{123} + 45$ is even, then x is odd.

Proof. We will prove the contrapositive:

If x is even, then $x^{2020} - 3 \cdot x^{123} + 45$ is odd.

Assume x is even. Then $x^{2020}, -3 \cdot x^{123}$ is even.

Thus, $x^{2020} - 3 \cdot x^{123}$ is even.

Since 45 is odd, then

$x^{2020} - 3 \cdot x^{123} + 45$ is odd.

□